Technical Comments

Comments on "An Efficient, Steady, Subsonic Collocation Method for Solving Lifting-Surface Problems"

James D. Revell*
Lockheed-California Company, Burbank, Calif.

THIS Comment concerns the Paper[12] by A. M. Cunningham which deals with primarily one aspect of subsonic lifting-surface theory, namely the impact of accuracy of the numerical evaluation of the chordwise integral of the product of (pressure) × (kernel function), using the Mehler-Gaussian quadrature technique, which takes account of the form of the leading edge square root singularity in lifting pressure as a weighting function. The introduction[12] makes only cursory reference to a few other subsonic lifting surface methods and then plunges into the development of a refinement to the methods of Hsu and Rowe.[1,2]

The best known work on the subsonic kernel function procedure is that of Watkins, Woolston, and H. J. Cunningham,[3] based on the careful analytical investigation[4] of the singularities of both the steady and oscillatory kernel function.

The "NASA kernel function method"[3] evaluates the so-called Hadamard "finite part" of the singular spanwise integral of $G(\eta,y)/(y - \eta)^2$, by using polynomial fit to $G(\eta,y)$ in the neighborhood $\eta \rightarrow y$. The NASA method of Ref. 3 has been further developed at Lockheed, for example, with satisfactory and efficient results. It has never been conclusively proven that Hsu-Rowe method[1,2] is superior to the Ref. 3 method. It is believed that Refs. 3 and 4 should be cited and discussed, since these methods are widely used in industry in aeroelastic applications.

There are also many other contributions to this subject which deserve mention and citation. With respect to the question of integration of the singularities and choices of loading functions, the works of Laschka[5] and La Clerc[6] and Wagner[7] are pertinent. Also, an excellent review of European as well as American work and various alternative formulations was given by Landahl and Stark.[8] In regard to the efficiency of collocation methods, Stark[4] should be cited and discussed.

Aside from the numerical evaluation of the product of (kernel function) × (loading function), other aspects enter into the over-all question of accuracy and efficiency. By way of example, Rodden and Revell[10,11] have discussed various choices of loading functions and the possibility of using a "least squared error" vs the collocation method of solving the lifting-surface problem when the downwash distribution on the surface is complicated. In such cases, it might be preferable to satisfy the downwash conditions at many sample points on the wing (in a least squares sense) rather than exactly matching downwash at a relatively few points. The recent generation of computers has made possible the direct matrix inversions necessary for collocation problems of 100 or more points, so that the direct collocation methods now may provide enough sampling to supersede the least squares approach, at least for most airloads purposes.

Cunningham's paper is somewhat misleading, in regard to accuracy. For example, for purposes of pressure predictions for wing design, 2D airfoil theory experience suggests that one needs, say, of the order of 20–25 chordwise points, including points very close to the leading edge (say within $\frac{1}{2}\%$ chord), to accurately evaluate the effects of camber shape near the edges. In such cases, lifting-surface solutions, of the kind discussed in Ref. 12, involving 4 chordwise downwash points per chord with 4 chordwise loading functions, simply do not provide a close enough definition of the effect of camber on local pressure, despite the accuracy of evaluation of the chordwise integral of the product (kernel function) × (loading function).

In summary, Cunningham's paper does make a worthwhile contribution to the important question of numerical evaluation of the integral over the surface of the product (kernel) (assumed loading function); however, there are many other important aspects of the total problem which are ignored in this paper and which have received much attention elsewhere, especially in the unsteady aerodynamics and aeroelastic literature,[3–11] about which the author should be made aware. Almost any paper on lifting-surface theory nowadays must have a very good literature critique in its introduction, if it is to be worthy of consideration because of the proliferation of papers on lifting-surface theory. This writer believes that Refs. 3–11, and possibly others mentioned in Ref. 8, should be cited with at least a brief remark classifying the aspects of the problem discussed in those papers and emphasizing that Cunningham's paper deals primarily with a particular aspect, albeit an important one. This writer believes that the present paper,[12] with the added discussion of the other aspects previously mentioned, does make a worthy contribution to the literature, because many companies spend considerable sums on achieving efficient lifting-surface procedures, and it is especially important to classify the many different approaches which have been published to avoid future duplicative research efforts.

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*R & D Scientist, Research and Methods, Aero-Propulsion Department, Flight Sciences Division, Science and Engineering Branch. Associate Fellow AIAA.

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Reply by Author to J. D. Revell

A. M. Cunningham Jr*
General Dynamics Corporation, Fort Worth Division, Fort Worth, Texas

The author wishes to thank Revell for pointing out areas which require further discussion. These areas are concerned with the relationship of the present method with other methods, types of pressure functions, and the minimum number of downwash points (or pressure functions) required for solutions of acceptable accuracy.

The purpose of the research partly summarized in this paper is to develop an aerodynamic prediction technique which combines the efficiency of Hsu's method with the accuracy of the NASA method[1] or Laschka's method.[2] In particular, this paper is concerned with the instabilities of Hsu's method and the refinements that are necessary to remove them. This paper also represents the first of two papers which, together, present a method of accurately predicting airloads on arbitrary planar surfaces in steady or oscillatory subsonic flow with minimum computational effort.

As was pointed out in the summary, introduction, and conclusions, the paper had several other major objectives besides discussing the importance of chordwise integration and developing a more accurate means of performing this step. One of these was to establish a criterion for choosing, a priori, a minimum number of downwash points, and, hence, pressure functions, that would yield a solution of acceptable accuracy. Another objective was to show that the present method was more efficient than finite-element methods for predicting aerodynamic loadings on planar surfaces of arbitrary planform.

The present method is compared with several other classic methods in this paper and is shown to be equal or superior in accuracy. In a following paper[3] on the oscillatory version of this method, it is also shown to be equal in accuracy to Laschka's method on planar surfaces without discontinuities in planform geometry. This constitutes an indirect comparison with the NASA method developed by Watkins, Woolston, and H. J. Cunningham,[2] since Laschka has shown that his method yields essentially same results as NASA method.

The pressure functions used in the present solution, however, are not the best. For the chordwise functions, it is felt by the author that those used by Laschka[2] and Wagner[4] might be the best. For the spanwise functions, the present method has been modified to use Tschebyshev polynomials of the second kind $U_n(\mathbf{x})$ as defined in Ref. 5, in lieu of the simple polynomials as given in Eq. (3b). This modification yields matrices which are far better behaved than those obtained for the simple polynomial type of spanwise functions. As a result, the method has been applied successfully to surfaces with strong discontinuities in the local chord such as those encountered in planar representations of wing-body or wing-tip store configurations.

The most important aspect questioned by Revell is the choice of a suitable number of downwash points and the corresponding pressure functions. The basic question here is how well the chosen downwash points define the true downwash distribution. This all depends on the spacing of the points in both the chordwise and spanwise directions.

If an even spacing of points is assumed, this effectively results in a definition of the downwash at all points between the control points with a Taylor Series expansion of the distribution. Such expansions have the slowest possible convergence,[5] and the resulting interpolation functions can diverge even for very smooth downwash functions.[6] Thus the use of evenly spaced downwash points can lead to pressure function solutions whose induced downwash functions oscillate violently between the downwash points. In this case, a least-squares approach would provide some stability, but it could never yield the best fit, which is unique.[6]

If an uneven spacing is used, specifically the $\mathbf{x}_i$ as given in Eq. (19) in the chordwise direction, the quality of the curve fitting is vastly improved. The use of these points yields an interpolated function with accuracy and smoothness that approaches the best fit possible in the least-squares sense without the difficulties inherent in arbitrarily choosing a least-squares solution. The $\mathbf{x}_i$, for $i = 1, 2, \ldots, \bar{m}$, are the $\bar{m}$ zeros of a polynomial $Q_{\bar{m}}(\mathbf{x})$ defined as $Q_0(\mathbf{x}) = 1, Q_{\bar{m}}(\mathbf{x}) = [U_{\bar{m}}(\mathbf{x}) - U_{\bar{m}+1}(\mathbf{x})]$, $\bar{m} > 0$, where $U_n(x)$ are Tschebyshev polynomials of the second kind. The $\mathbf{x}_i$ were originally chosen by Hsu to yield zero error in the predicted sectional lift if the chordwise downwash variation is of degree $(2\bar{m} - 1)$ or less. [The generalized force is exact if the downwash is of degree $(\bar{m} - 1)$ or less.] This consideration was made for average lift; however, something can also be said for what happens over the entire surface. Since the set of $\mathbf{x}_i$ is the zeros of $Q_m(x)$, then it can be shown that the error in the induced downwash, $\Delta w(\mathbf{x})$, for $\bar{m} \geq 3$ is approximately given as $\Delta w(x) \approx a_{\bar{m}} U_{\bar{m}}(\mathbf{x}) + a_{\bar{m}+1} U_{\bar{m}+1}(\mathbf{x}) + a_{\bar{m}+2} U_{m+2}\mathbf{x}) + \ldots$, where the a_n are the coefficients of the $n \geq \bar{m}$ terms of a $U_n(\mathbf{x})$ polynomial expansion of the exact downwash function. Thus one may establish approximate error bounds on the induced downwash at points other than the control points by the $\Delta w(\mathbf{x})$ given previously. Although a $U_n(\mathbf{x})$ expansion does not converge quite as rapidly as an expansion of Tschebyshev polynomials of the first kind, $T_n(\mathbf{x})$, it is far more rapid than a Taylor series expansion.

As an estimate of the error involved, the ratio of the $a_{\bar{m}}$ coefficient from the previous equation to the $b_{\bar{m}}$ coefficient of a Taylor series expansion of the same function is approximately[5] $a_{\bar{m}}/b_{\bar{m}} \approx (\bar{m} + 1)/2^{\bar{m}}$, Thus for $\bar{m} = 4$, $a_4/b_4 = \frac{5}{16}$; and for $\bar{m} = 5$, $a_5/b_5 = \frac{6}{32}$. For the $c_{\bar{m}}$ coefficient of a $T_n(x)$ expansion of the same function, the ratio is $c_{\bar{m}}/b_{\bar{m}} \approx 1/2^{\bar{m}-1}$, which yields for $\bar{m} = 4$, $c_{\bar{m}}/b_{\bar{m}} = \frac{1}{8}$, and for $\bar{m} = 5$, $c_{\bar{m}}/b_{\bar{m}} = \frac{1}{16}$. Thus if the degree of the camber is known, then an almost exact fit may be achieved. If the degree is not known, then, because of rapid convergence of $U_n(\mathbf{x})$ expansion, the fit will approach best fit as afforded by $T_n(\mathbf{x})$ expansion.

More specifically, experience has shown that it is sufficient to use a number of chordwise points, or pressure functions, that are a minimum of three or equal to two plus the estimated degree of the camber line variation. Then by use of the ratio defined by Eq. (24) to determine the number of spanwise terms, pressure distributions are predicted which show excellent correlation with local measured distributions.

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* Senior Structures Engineer. Member AIAA.

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